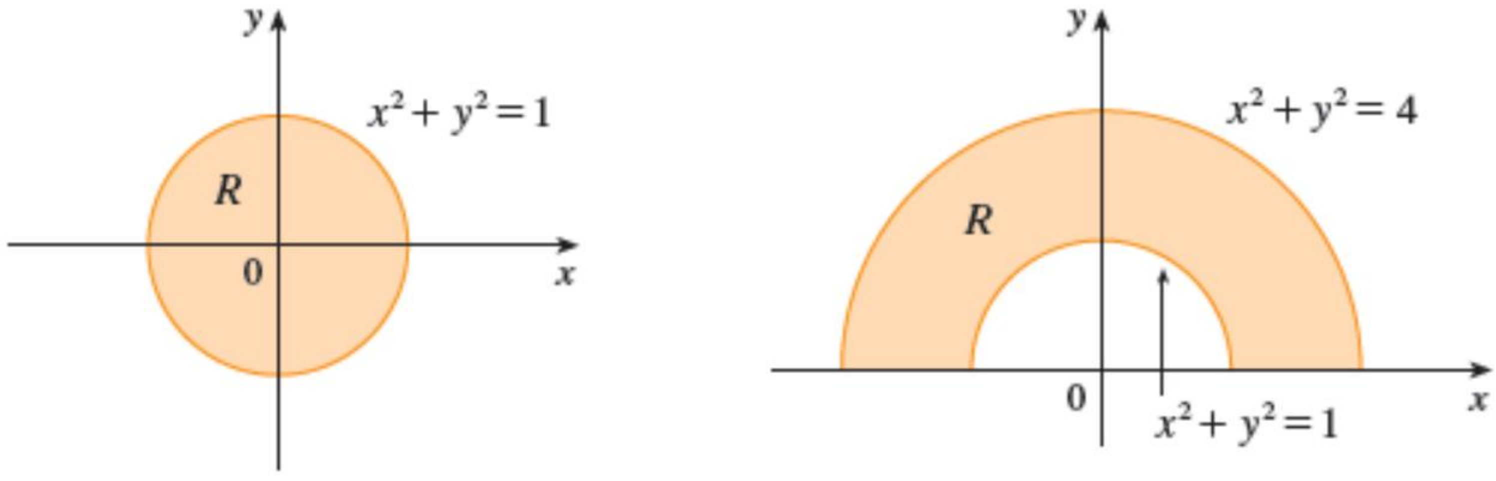
Section 15.3: Double Integrals In Polar Coordinates

What We'll Learn In Section 15.3

1. Double Integrals In Polar Coordinates

1. Double Integrals In Polar Coordinates

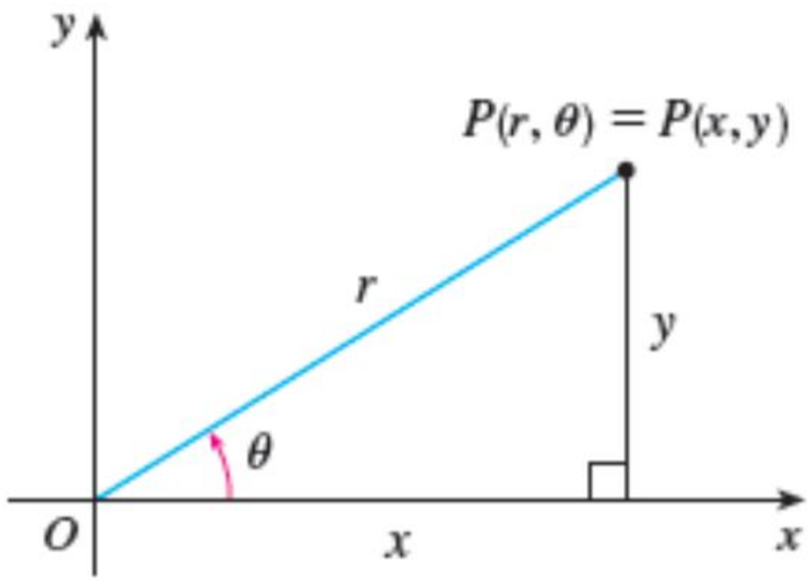
When calculating a double integral of a 2-variable function $f(x, y)$ over a region D, if the region you're integrating over is somewhat circular, using polar coordinates to do the integral may make the integral simpler to calculate.



(a) $R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

1. Double Integrals In Polar Coordinates

<u>Recall</u>: Stuff about polar coordinates…



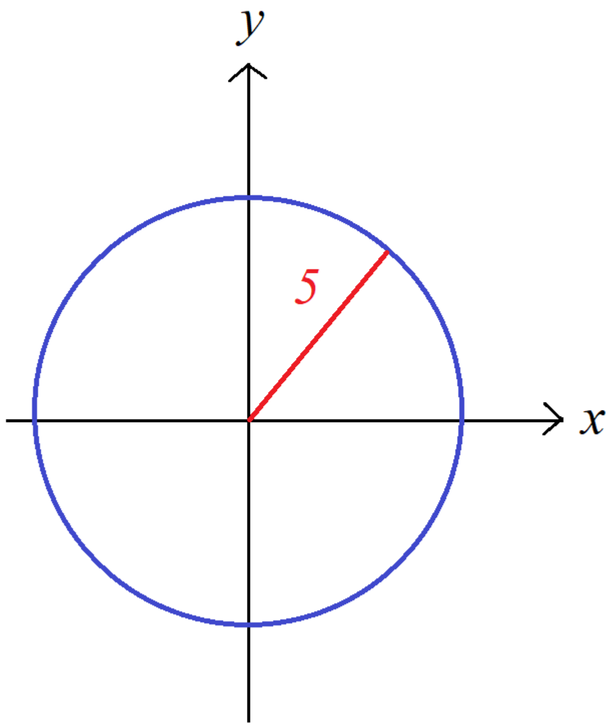
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r^2 = x^2 + y^2$$

1. Double Integrals In Polar Coordinates

Recall: Stuff about polar coordinates…

Equations of circles with the origin as the center are easy in polar coordinates…

$r = \#$ is the equation of a circle
center = origin
radius = #
$x^2 + y^2 = 25$ becomes $r = 5$



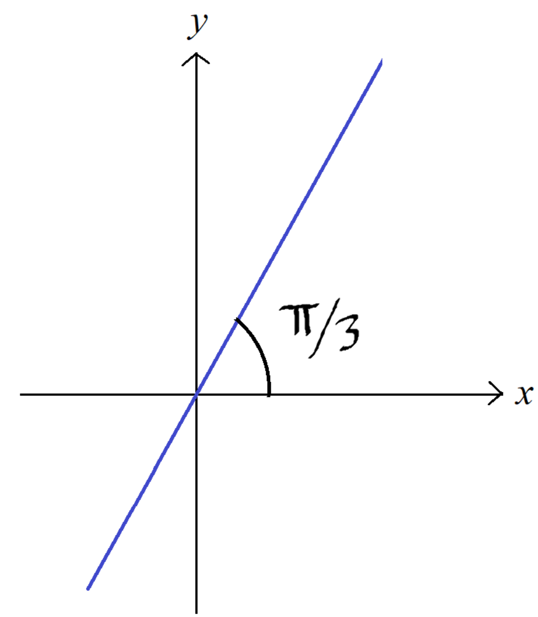
1. Double Integrals In Polar Coordinates

Recall: Stuff about polar coordinates…

Equations of lines through the origin are easy in polar coordinates…

$\theta = \#$ is the equation of a line through the origin

$y = \sqrt{3}x$ becomes $\theta = \pi/3$



1. Double Integrals In Polar Coordinates

<u>The polar rectangle (subrectangles)</u>

Calculating double integrals in rectangular coordinates or in polar coordinates always gives the same result.

Talk about slicing in rectangular coordinates.
Talk about slicing in polar coordinates.

Just instead of using $\Delta x \Delta y$ (or $dxdy$),
use the area of a polar subrectangle $r_i \Delta r \Delta \theta$ (or $rdrd\theta$)

1. Double Integrals In Polar Coordinates

<u>The polar rectangle (subrectangles)</u>

Calculating double integrals in rectangular coordinates or in polar coordinates always gives the same result.

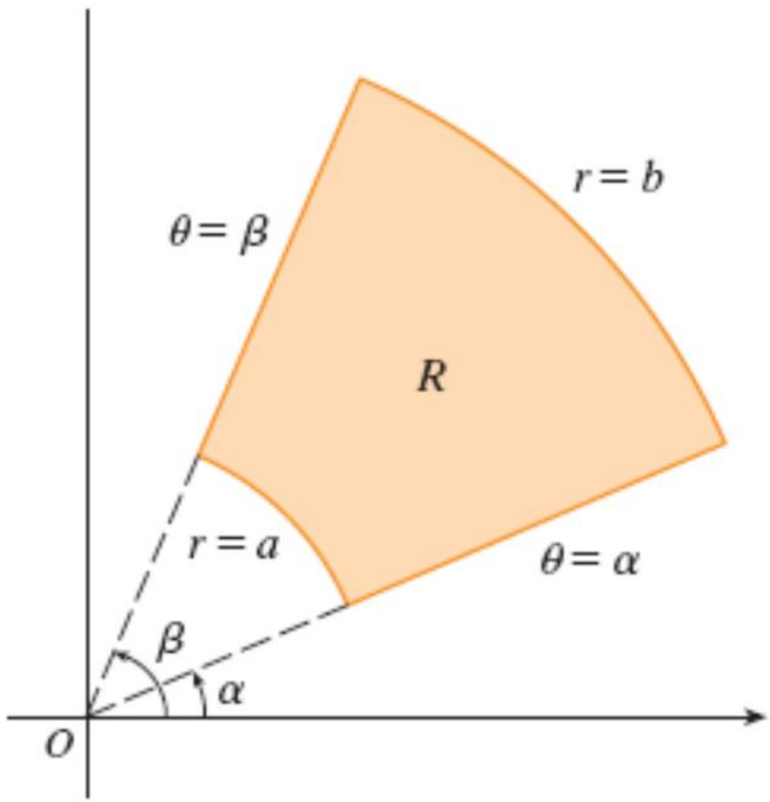
Talk about slicing in rectangular coordinates.
Talk about slicing in polar coordinates.

Just instead of using $\Delta x \Delta y$ (or $dxdy$),
use the area of a polar subrectangle $r_i \Delta r \Delta \theta$ (or $r dr d\theta$)

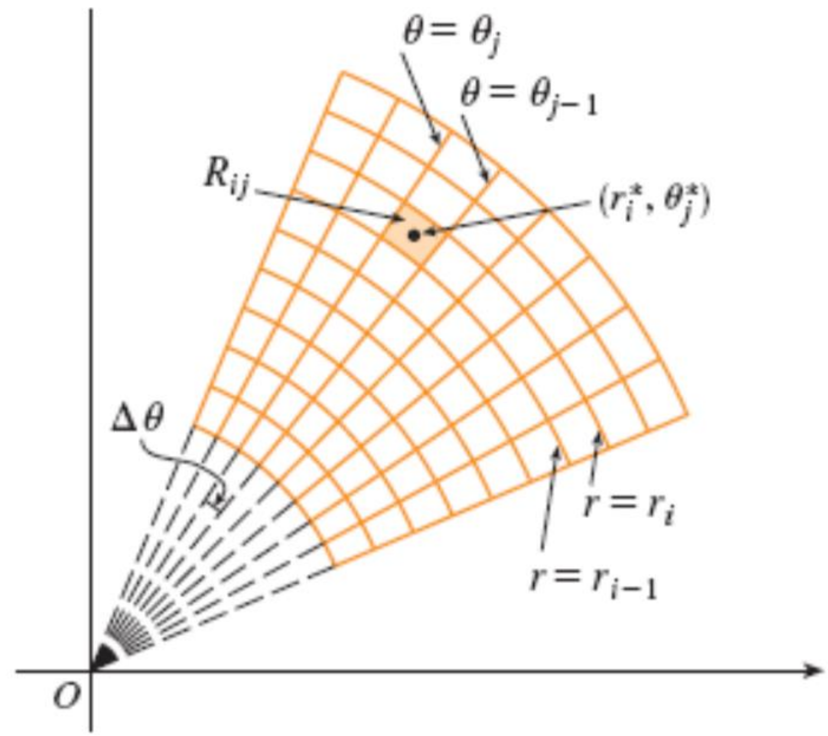
1. Double Integrals In Polar Coordinates

<u>The polar rectangle (subrectangles)</u>

Polar rectangle



Dividing R into polar subrectangles



1. Double Integrals In Polar Coordinates

The calculate a double integral in polar coordinates…

1. Replace all x's with $r\cos\theta$
2. Replace all y's with $r\sin\theta$
3. Replace dA with $rdrd\theta$
4. Write the integration limits using polar numbers/functions

1. Double Integrals In Polar Coordinates

The calculate a double integral in polar coordinates…

2 Change to Polar Coodinates in a Double Integral

If f is continuous on a polar rectangle R given by

$0 \leqslant a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta,$ where $0 \leqslant \beta - a \leqslant 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

1. Double Integrals In Polar Coordinates

Ex 1:

Evaluate $\iint_R 3x + 4y^2 \, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

1. Double Integrals In Polar Coordinates

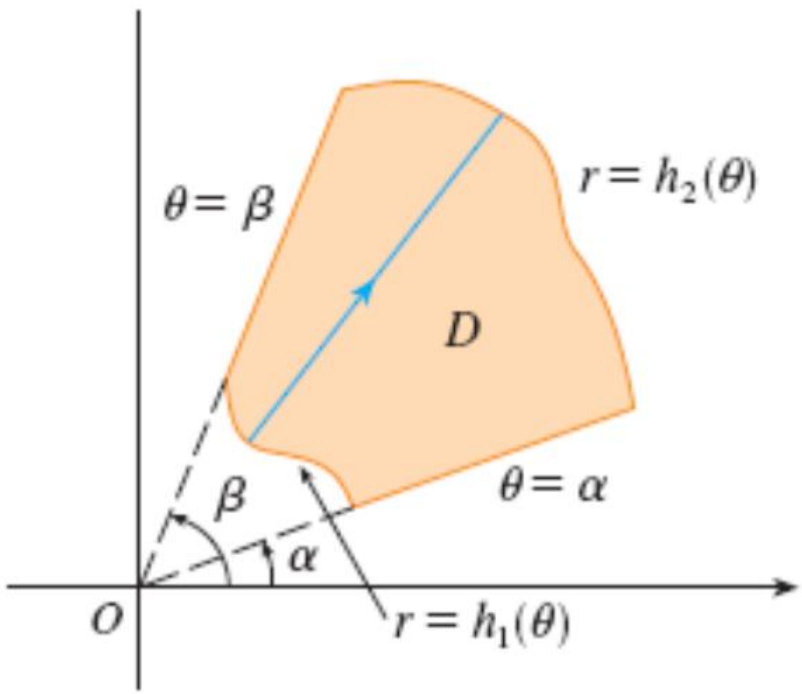
Ex 2:

Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

1. Double Integrals In Polar Coordinates

<u>The analogue to type I and II regions in polar coordinates:</u>

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$



1. Double Integrals In Polar Coordinates

The analogue to type I and II regions in polar coordinates:

If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

Then

$$\iint\limits_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

1. Double Integrals In Polar Coordinates

<u>Ex 3:</u>

Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$$A(D) = \iint\limits_{D} 1\, dA = \int_{\alpha}^{\beta} \int_{0}^{h(\theta)} r\, dr\, d\theta$$

$$= \int_{\alpha}^{\beta} \left[\frac{r^2}{2}\right]_0^{h(\theta)} d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left[h(\theta)\right]^2 d\theta$$

1. Double Integrals In Polar Coordinates

Ex 3:

Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

1. Double Integrals In Polar Coordinates

Ex 4:

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the *xy*-plane, and inside the cylinder $x^2 + y^2 = 2x$.